SOLUTION OF SOME PROBLEMS OF ADJOINING BY REDUCTION TO THE GENERALIZED RIEMANN PROBLEM

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In this work, we understand by the problem of adjoining, the problem of finding in some two-dimensional region D_1 the solution of the equation $\operatorname{div}(k \operatorname{grad} p) = 0$ under the condition that the coefficient k is a piece-wise continuous function of the x, y coordinates. On the boundaries γ where the coefficient k is discontinuous the following conditions of adjoining must be fulfilled

$$p_+ = p_-, \qquad k_+\left(\frac{\partial p}{\partial n}\right)_+ = k_-\left(\frac{\partial p}{\partial n}\right)_- \tag{I}$$

Here the subscripts minus and plus denote the limiting values of the function at the boundaries γ of discontinuity; n is the normal to γ. The solution function $p(x, y)$ has singularities of the logarithmic type in the region D, and if D has a boundary Γ, the function $p(x, y)$ must satisfy on the Γ boundary conditions of the first, second, or third kind.

This problem has many applications. For example, in underground hydromechanics a function p, which satisfies the conditions given above, determines the pressure field in a piece-wise inhomogeneous oil-containing layer subjected to water pressure and to the linear law of filtration; in the electromagnetic theory, the function p yields the statistical distribution of temperature in a piece-wise nonuniform heat conducting medium. The proposed problem is reduced by the method presented in [1] and [2] to a singular integral equation (or a system of equations) for the determination of the solution function $p(x, y)$ on γ. The resulting equation is then in turn reduced to the generalized Riemann problem [4] by the use of the properties of integrals of the Cauchy-Hadamard type. The solution of the obtained generalized Riemann problem can then be easily obtained in a number of cases. Among the concrete problems of adjoining considered in this work are the following:

1. The region D is the entire plane. The boundary γ is the real axis. It divides D into two subregions: the upper half-plane D_+, in which k takes on the value k_1 = const, and the lower half-plane D_-, in which k takes on a value k_2 = const. At a certain point $A(x_0, y_0) \in D_+$, the function p has a logarithmic singularity.

2. The region D is the entire plane; the boundary γ is the circumference of a circle of unit radius. The region D_+ is the interior of this circle where $k = k_1$ = const, and p has a logarithmic singularity at a point $A(r_0, \theta_0)$; D_- is the exterior of this circle, where $k = k_2$ = const.

3. The region D is the interior of a circle of unit radius; γ is the diameter of this circle which coincides with the real axis. In the region D_+, given by ($|z| \leqslant 1$, Im $z > 0$, $z = x + iy$), the coefficient k is $k_1 = (b_1 y + c_1)^2$, and the function p has a singularity at the point $A(x_0, y_0)$. In the region D_- given by ($|z| \leqslant 1$, Im $z < 0$) the coefficient k is $k_2 = (b_2 y + c_2)^2$. When $|z| = 1$, the function $p = 0$.

4. The region D is a rectangle: $-\alpha \leqslant x \leqslant \delta$, $-\beta \leqslant y \leqslant \beta$. The equation of the boundary γ is $x = \alpha$. In the region D_+ ($-\alpha \leqslant x < \alpha$, $-\beta \leqslant y \leqslant \beta$) the coefficient k is $k_1 = (a_1 + b_1 x + c_1 y)^2$ and $p(x, y)$ has a singularity at the point $A(x_0, y_0)$. In the region D_-($\alpha < x \leqslant \delta$, $-\beta \leqslant y \leqslant \beta$) the coefficient k is $k_2 = (a_2 + b_2 x + c_2 y)^2$. On the boundary of D the function $p = 0$.

The first two of these problems have been solved before by methods different from ours. Their inclusion in this article makes it possible for us to demonstrate the usefulness and correctness of our new approach to the solution of problems of adjoining by comparing our results with those obtained earlier.

Finally, we shall show that the problems of adjoining arise in many areas of physics and mechanics such as the theories of electromagnetism, heat conduction, underground hydromechanics and others.

Let us consider the first one of the four mentioned problems.

We shall seek a function $p(x, y)$ in the regions D_+ and D_- of the forms

$$p_+(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} p(t) \frac{y\,dt}{(t-x)^2 + y^2} + G \qquad (2)$$

$$p_-(x, y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} p(t) \frac{y\,dt}{(t-x)^2 + y^2} \qquad (3)$$

respectively. Here $p(t)$ stands for the unknown function on the real axis; G is Green's function for the upper half-plane

$$G = \frac{Q}{4\pi k_1} \ln \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \qquad (4)$$

The quantity Q is the intensity of the source at the point $A(x_0, y_0)$. The solution of the assumed form satisfies the first one of the conditions (1) identically; the second one of these conditions yields the following integral equation for the determination of $p(t)$:

$$\int_{-\infty}^{\infty} p(t) \frac{dt}{(t-x)^2} = \frac{Q}{k_1 + k_2} \frac{y_0}{(x-x_0)^2 + y_0^2} \qquad (5)$$

For the solution of this equation we introduce the piece-wise analytic function

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} p(t) \frac{dt}{t-z}$$

The limiting values of this function and of its derivative have the form

$$\Phi^{\pm}(x) = \pm \frac{p(x)}{2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{p(t)\,dt}{t-x}, \qquad \Phi'^{\pm}(x) = \pm \frac{p'(x)}{2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{p(t)\,dt}{(t-x)^2} \qquad (6)$$

Here, the integral in the first formula is interpreted to mean the Cauchy principal value, while in the second formula it is taken in the Cauchy-Hadamard sense. With the aid of (6), equation (5) is reduced to the following Riemann problem:

$$\Phi'^{+}(x) + \Phi'^{-}(x) = \frac{Q y_0}{\pi i (k_1 + k_2)} \frac{1}{(x-x_0)^2 + y_0^2} \qquad (7)$$

or, with the introduction of a new function $\Phi_1(z)$, to the form

$$\Phi_1^{+} - \Phi_1^{-} = \frac{Q y_0}{\pi i (k_1 + k_2)} \frac{1}{(x-x_0)^2 + y_0^2}$$

This is the simplest problem of determining a function by its discontinuity. The solution is given by an integral of the Cauchy type

$$\Phi_1(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{Qy_0 dt}{\pi i (k_1+k_2)\,[(t-x_0)^2+y_0^2]\,(t-z)}$$

Let us denote the density of this integral by $\varphi(t)$.

Suppose $\varphi(t) = \varphi^+(t) + \varphi^-(t)$, where $\varphi^+(t)$ and $\varphi^-(t)$ are the boundary values of functions that are analytic in D_+ and D_-. Then, on the basis of Cauchy's formula we have

$$\Phi_1^+(z) = \varphi^+(z) + \varphi^-(\infty), \qquad \Phi_1^-(z) = -\varphi^-(z) + \varphi^-(\infty)$$

Since

$$\varphi(t) = \frac{Qy_0}{\pi i (k_1+k_2)}\frac{1}{(t-x_0)^2+y_0^2} = -\frac{Q}{2\pi i (k_1+k_2)}\left[\frac{i}{t-x_0-iy_0} - \frac{i}{t-x_0+iy_0}\right]$$

then

$$\Phi_1^+(z) = \frac{Q}{2\pi (k_1+k_2)}\frac{1}{z-\bar{z}_0}, \qquad \Phi_1^-(z) = \frac{Q}{2\pi (k_1+k_2)}\frac{1}{z-z_0}$$

and, since $\Phi'^+ = \Phi_1^+$, $\Phi'^- = -\Phi_1^-$, we find, by formula (6), that

$$p(x) = \int \frac{Q}{\pi (k_1+k_2)}\frac{(x-x_0)\,dx}{(x-x_0)^2+y_0^2} = \frac{Q}{2\pi (k_1+k_2)}\{\ln[(x-x_0)^2+y_0^2]+c_1\}$$

Thus, on the basis of the formulas (2), and (3), we have

$$p_*^\pm(x,y) = \pm\frac{Q}{2\pi (k_1+k_2)}\,\mathrm{Im}\,\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2i\{\ln[(t-x_0)^2+y_0^2]+c_1\}}{t-z}\,dt$$

In accordance with Cauchy's formula we obtain

$$p_*^\pm(x,y) = \frac{Q}{2\pi (k_1+k_2)}\ln[(x-x_0)^2+(y\pm y_0)^2]+c \tag{8}$$

$$p^+(x,y) = p_*^+(x,y)+G, \qquad p^-(x,y) = p_*^-(x,y) \tag{9}$$

Direct verification shows that the functions given by (8) and (9) satisfy the required conditions.

In the second problem it is convenient to use the notation

$$z = re^{i\theta}, \qquad z_0 = r_0e^{i\theta_0}, \qquad \zeta = e^{i\sigma}, \qquad \omega = e^{i\psi}$$

The solution is sought in the form

$$p^+(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} p(\sigma)\,\frac{1-r^2}{1-2r\cos(\sigma-\theta)+r^2}\,d\sigma + G \tag{10}$$

$$p^-(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} p(\sigma)\,\frac{r^2-1}{1-2r\cos(\sigma-\theta)+r^2}\,d\sigma - \frac{Q}{2\pi k_2}\ln r \tag{11}$$

where

$$G = \frac{Q}{4\pi k_1}\ln\frac{r^2r_0^2-2rr_0\cos(\theta-\theta_0)+1}{r_0^2-2rr_0\cos(\theta-\theta_0)+r^2}$$

The second condition of adjoining leads to the following integral equation for the determination of $p(\sigma)$

$$\frac{1}{4\pi}\int_0^{2\pi}\frac{p(\sigma)\,d\sigma}{\sin^2(\sigma-\psi)/2} = \frac{Q}{2\pi(k_1+k_2)}\left(\frac{r_0^2-1}{1-2r_0\cos(\psi-\theta_0)+r_0^2}+1\right) \tag{12}$$

The integral on the left-hand side is here understood in the Cauchy-Hadamard sense. For the solution of this equation we introduce a function $\Phi(z)$ and its derivative

$$\Phi(z) = \frac{1}{2\pi i}\oint p(\zeta)\left[\ln(\zeta-z)+\frac{\zeta}{\zeta-z}\right]d\zeta,\quad \Phi'(z) = \frac{1}{2\pi i}\oint p(\zeta)\frac{z\,d\zeta}{(\zeta-z)^2}$$

The limiting values of $\Phi'(z)$, as z approaches the circumference $|z| = 1$, are equal to

$$\Phi'^{\pm}(\omega) = \pm\frac{\omega p'(\omega)}{2} - \frac{1}{8\pi}\int_0^{2\pi} p(\sigma)\,\frac{d\sigma}{\sin^2(\sigma-\psi)/2}$$

Hence

$$\frac{1}{4\pi}\int_0^{2\pi} p(\sigma)\,\frac{d\sigma}{\sin^2[(\sigma-\psi)/2]} = -(\Phi'^{+}(\omega)+\Phi'^{-}(\omega)) \tag{13}$$

$$p(\omega) = \int\frac{\Phi'^{+}(\omega)-\Phi'^{-}(\omega)}{\omega}\,d\omega \tag{14}$$

The equation (12) is thus reduced to the Riemann problem

$$\Phi'^{+} + \Phi'^{-} = \frac{Q}{2\pi(k_1+k_2)}\left(\frac{1-r_0^2}{1-2r_0\cos(\psi-\theta_0)+r_0^2}-1\right)$$

Solving this problem, we obtain by means of (14) the function $p(\omega)$

$$p(\omega) = -\frac{Q}{2\pi(k_1+k_2)}\left[\ln\left(\omega-\frac{1}{z_0}\right)+\ln(\omega-z_0)-\ln\omega\right]+c_1$$

Finally, from (10) and (11) we obtain

$$p^+(r,\theta) = -\frac{Q}{\pi(k_1+k_2)}\ln\sqrt{r^2r_0^2-2rr_0\cos(\theta-\theta_0)+1}+G+c$$

$$p^-(r,\theta)=-\frac{Q}{\pi(k_1+k_2)}\left(\ln\sqrt{r^2-2rr_0\cos(\theta-\theta_0)+r_0^2}-\ln r\right)-\frac{Q}{2\pi k_2}\ln r+c$$

In the problems 3 and 4 we pass from the functions $p^{\pm}(s, y)$ to harmonic functions $u^{\pm}(x, y)$ by means of the substitutions

$$p^+\sqrt{k_1}=u^++\frac{Q}{2\pi\sqrt{k_0}}G, \qquad p^-\sqrt{k_2}=u^- \tag{15}$$

where k_0 is the value of the coefficient k_1 at the point $A(x_0, y_0)$.

The conditions of adjoining on γ for the functions $u^{\pm}$ take the form

$$u^+=\sqrt{\frac{k_+}{k_-}}u^- \qquad (u^{\pm}=0 \quad \text{on} \quad \Gamma) \tag{16}$$

$$\frac{\partial u^+}{\partial n}\sqrt{\frac{k_+}{k_-}}-\frac{\partial u^-}{\partial n}+\frac{Q}{2\pi\sqrt{k_0}}\frac{\partial G}{\partial n}\sqrt{\frac{k_+}{k_-}}=u^+\left(\frac{1}{\sqrt{k_-}}\frac{\partial\sqrt{k_+}}{\partial n}-\frac{1}{\sqrt{k_+}}\frac{\partial\sqrt{k_-}}{\partial n}\right) \tag{17}$$

In problem 3, the solution, which satisfies the conditions (16), is sought in the form

$$u^+(x,y)=\frac{1}{\pi}\int_{-1}^{1}yu(\tau)\left\{\frac{1}{(\tau-x)^2+y^2}-\frac{1}{(1-\tau x)^2+\tau^2y^2}\right\}d\tau \tag{18}$$

$$u^-(x,y)=-\frac{c_2}{c_1\pi}\int_{-1}^{1}yu(\tau)\left\{\frac{1}{(\tau-x)^2+y^2}-\frac{1}{(1-\tau x)^2+\tau^2y^2}\right\}d\tau \tag{19}$$

Condition (17) leads now to an integral equation for $u(x)$

$$u(x)-\frac{1}{\alpha\pi}\int_{-1}^{1}u(\tau)\left\{\frac{1}{(\tau-x)^2}-\frac{1}{(1-\tau x)^2}\right\}d\tau-\beta i\frac{\partial G(z,z_0)}{\partial z}\Big|_{y=0}=0 \tag{20}$$

Here

$$G(z,z_0)=\ln\frac{(z-z_0)(z-z_0^{-1})}{(z-\bar{z}_0)(z-\bar{z}_0^{-1})}$$

is Green's function for the semicircle

$$\alpha=\frac{b_1c_1-b_2c_2}{c_1^2+c_2^2}, \qquad \beta=\frac{Qc_1^2}{2(b_1c_1-b_2c_2)\pi\sqrt{k_0}} \qquad (b_1c_1-b_2c_2\neq 0)$$

Let us introduce the piece-wise analytic function

$$\Phi(z)=\frac{1}{2\pi i}\int_{-1}^{1}u(\tau)\left(\frac{1}{\tau-z}+\frac{1}{\tau-z^{-1}}-\frac{1}{\tau}\right)d\tau$$

and its derivative $\Phi'(z)$.

With the aid of these functions, equation (2) can be reduced to the generalized Riemann problem of finding a piece-wise analytic function from conditions given on the real axis

$$\Phi'^{+}(x) + i\alpha\Phi^{+}(x) = -\Phi'^{-}(x) + i\alpha\Phi^{-}(x) - \alpha\beta f(x) \qquad \text{for } |x| < 1$$

$$\Phi'^{+}(x) + \frac{i\alpha}{x^2}\Phi^{+}(x) = -\Phi'^{-}(x) + \frac{i\alpha}{x^2}\Phi^{-}(x) + \frac{\alpha\beta}{x^2} f\left(\frac{1}{x}\right) \qquad \text{for } |x| > 1$$

where

$$f(x) = \frac{2iy_0}{(x - x_0)^2 + y_0^2} - \frac{2iy_0(x_0^2 + y_0^2)}{[x(x_0^2 + y_0^2) - x_0]^2 + y_0^2}$$

Solving this problem, and making use of the formulas of Sokhotskii for the limiting values of the function Φ on γ, we find that, for $|x| < 1$

$$u(x) = F(x) - B_1 \cos\alpha x - B_2 \sin\alpha x$$

where

$$B_1 = \frac{F(1) + F(-1)}{2\cos\alpha}, \qquad B_2 = \frac{F(1) - F(-1)}{2\sin\alpha}$$

$$F(x) = 2\alpha\beta\left\{e^{\alpha y_0}\left[\cos\alpha(x - x_0)\ln\alpha\sqrt{(x - x_0)^2 + y_0^2} - \sin\alpha(x - x_0)\tan^{-1}\frac{x - x_0}{y_0}\right] - \right.$$

$$- \exp\frac{\alpha y_0}{x_0^2 + y_0^2}\left[\cos\alpha(x - x_0)\ln\alpha\sqrt{(x - x_0)^2 + \frac{y_0^2}{(x_0^2 + y_0^2)^2}} - \right.$$

$$\left. - \sin\alpha(x - x_0)\tan^{-1}\frac{(x - x_0)(x_0^2 + y_0^2)}{y_0}\right] +$$

$$+ e^{\alpha y_0}\sum\frac{(-1)^n\alpha^n\left(\sqrt{(x - x_0)^2 + y_0^2}\right)^n}{n\cdot n!}\cos\left[\alpha(x - x_0) + n\tan^{-1}\frac{x - x_0}{y_0}\right] -$$

$$- \exp\frac{\alpha y_0}{x_0^2 + y_0^2}\sum\frac{(-1)^n\alpha^n\left(\sqrt{(x - x_0)^2 + y_0^2/(x_0^2 + y_0^2)^2}\right)^n}{n\cdot n!} \times$$

$$\left. \times \cos\left[\alpha(x - x_0) + n\tan^{-1}\frac{(x - x_0)(x_0^2 + y_0^2)}{y_0}\right]\right\}$$

Here, the summation on n (and later on m) is performed from 1 to ∞. Omitting the details, which are analogous to the preceding ones, we now give the result for the fourth problem

$$u^{+}(x, y) = \frac{4Q(c + c_1 y)}{\alpha^2\beta\pi(a_1 + b_1 x_0 + c_1 y_0)}\sum\left\{\frac{\sinh[n\pi(\alpha + x)/2\beta]}{\sinh(n\pi\alpha/\beta)} \times\right.$$

$$\times \frac{\sin[n\pi(y + \beta)/2\beta]\sin[n\pi(y_0 + \beta)/2\beta]}{\left[(c + c_1 y)\coth\frac{n\pi\alpha}{\beta} + \frac{(d + c_2 y)^2}{c + c_1 y}\coth\frac{n\pi(\delta - \alpha)}{2\beta}\right]\frac{n}{2\beta} - \frac{e + fy}{(c + c_1 y)\pi}} \times$$

$$\times \sum \frac{(-1)^m m}{m^2 / \alpha^2 + n^2 / \beta^2} \sin \frac{m\pi (x_0 + \alpha)}{2\alpha} \Big\}$$

$$u^-(x, y) = \frac{4Q (d + c_2 y)}{\alpha^2 \beta \pi (a_1 + b_1 x_0 + c_1 y_0)} \sum \left\{ \frac{\sinh [n\pi (\delta - x) / 2\beta]}{\sinh [n\pi (\delta - \alpha) / 2\beta]} \times \right.$$

$$\times \frac{\sin [n\pi (y + \beta) / 2\beta] \sin [n\pi (y_0 + \beta) / 2\beta]}{\left[(c + c_1 y) \coth \frac{n\pi\alpha}{\beta} + \frac{(d + c_2 y)^2}{c + c_1 y} \coth \frac{n\pi (\delta - \alpha)}{2\beta}\right] \frac{n}{2\beta} - \frac{e + fy}{(c + c_1 y)\pi}} \times$$

$$\times \sum \frac{(-1)^m m}{m^2 / \alpha^2 + n^2 / \beta^2} \sin \frac{m\pi (x_0 + \alpha)}{2\alpha} \Big\}$$

where

$$c = a_1 + b_1 \alpha, \qquad d = a_2 + b_2 \alpha, \qquad e = c b_1 - d b_2, \qquad f = b_1 c_1 - b_2 c_2$$

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